

(2)

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$$1 * \lim_{x \rightarrow a} f(x) = b$$

$$* \varepsilon > 0 \exists \delta_{(\varepsilon)} > 0 \Rightarrow \forall x \in D_f : 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon$$

$$2 * \lim_{x \rightarrow a} f(x) = \infty$$

$$* M > 0 \exists \delta_{(M)} > 0 \Rightarrow \forall x \in D_f : 0 < |x - a| < \delta \Rightarrow f(x) > M$$

$$3 * \lim_{x \rightarrow a} f(x) = -\infty$$

$$* M > 0 \exists \delta_{(M)} > 0 \Rightarrow \forall x \in D_f : 0 < |x - a| < \delta \Rightarrow f(x) < -M$$

$$4 * \lim_{x \rightarrow \infty} f(x) = b$$

$$* \varepsilon > 0 \exists K_{(\varepsilon)} > 0 \Rightarrow \forall x \in D_f : x > K \Rightarrow |f(x) - b| < \varepsilon$$

$$5 * \lim_{x \rightarrow -\infty} f(x) = b$$

$$* \varepsilon > 0 \exists K_{(\varepsilon)} > 0 \Rightarrow \forall x \in D_f : x < -K \Rightarrow |f(x) - b| < \varepsilon$$

$$6 * \lim_{x \rightarrow \infty} f(x) = \infty$$

$$* M > 0 \exists K_{(M)} > 0 \Rightarrow \forall x \in D_f : x > K \Rightarrow f(x) > M$$

$$7 * \lim_{x \rightarrow \infty} f(x) = -\infty$$

$$* M > 0 \exists K_{(M)} > 0 \Rightarrow \forall x \in D_f : x > K \Rightarrow f(x) < -M$$

$$8 * \lim_{x \rightarrow -\infty} f(x) = \infty$$

$$* M > 0 \exists K_{(M)} > 0 \Rightarrow \forall x \in D_f : x < -K \Rightarrow f(x) > M$$



$$\lim_{x \rightarrow 3} (2x-5) = 1$$

لا يوجد كثير الحدود

الحل

$$1) f(x) = 2x-5, D_f = \mathbb{R} \Rightarrow$$

لا يوجد كثير الحدود

2)

نطلبوا ابتداءً

$$\forall \epsilon > 0 \exists \delta(\epsilon) > 0 \ni \forall x \in \mathbb{R} : 0 < |x-3| < \delta$$

$$\Rightarrow |2x-5-1| < \epsilon$$

$$|2x-6| < \epsilon$$

إذا أعطينا  $\epsilon > 0$  فاول ايجاد  $\delta > 0$  مناسب

$$\delta = \frac{\epsilon}{2}$$

$$|x-3| < \frac{\epsilon}{2} \Rightarrow 2|x-3| < \epsilon \Rightarrow |2x-6| < \epsilon$$

هامش

$$2x-6 < \epsilon$$

$$2|x-3| < \epsilon$$

$$|x-3| < \frac{\epsilon}{2}$$

$$\lim_{x \rightarrow -2} (2x^2-5) = 3$$

الحل

$$f(x) = 2x^2-5, D_f = \mathbb{R}$$

لا يوجد كثير الحدود

نطلبوا ابتداءً

$$\forall \epsilon > 0 \exists \delta(\epsilon) > 0 \ni \forall x \in D_f : 0 < |x+2| < \delta \Rightarrow$$

$$|2x^2-5-3| < \epsilon$$

$$|2x^2-8| < \epsilon$$

إذا أعطينا  $\epsilon > 0$  فاول ايجاد  $\delta > 0$  مناسب

نقرر  $\delta \leq 1$

$$\delta \leq 1 \Rightarrow -2-\delta < x < -2+\delta$$

الحل

هامش

$$|2x^2-8| < \epsilon$$

$$2|x^2-4| < \epsilon$$

$$2|x-2||x+2| < \frac{\epsilon}{2}$$

$$\Rightarrow -3 < x < -1 \Rightarrow |x| > 3$$

$$-5 < x-2 < -3 \Rightarrow |x-2| > 3$$

$$|x-2| \leq |x| + |2| < 5 \quad \delta = \frac{\epsilon}{10}$$

$$0 < |x+2| < \delta < \frac{\epsilon}{10} \Rightarrow 10|x+2| < \epsilon$$

$$\Rightarrow (2)(5) |x+2| < \epsilon \Rightarrow 2|x-2||x+2| < \epsilon$$

$$\Rightarrow 2|x^2-4| < \epsilon \Rightarrow |2x^2-8| < \epsilon$$

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$$\lim_{x \rightarrow 3} (4 - x^3) = -23$$

1. حل

$$f(x) = 4 - x^3 \quad \text{و} \quad D_f = \mathbb{R} \quad \text{دالة كثيرة حدود}$$

$$\forall \varepsilon > 0 \exists \delta_{\varepsilon} > 0 \Rightarrow \forall x \in D_f \Rightarrow 0 < |x - 3| < \delta \Rightarrow$$

$$\Rightarrow |4 - x^3 + 23| < \varepsilon$$

$$|27 - x^3| < \varepsilon$$

2. إيجاد  $\delta$

قاعدة إيجاد علاقة بين المقدارين  $|x - 3|$  و  $|27 - x^3|$

$$|27 - x^3| = |x^3 - 27| = |x - 3| |x^2 + 3x + 9| \quad \text{--- 1.}$$

$$\exists 3 - \delta, 3 + \delta \quad \text{و} \quad \delta \leq 7$$

$$2 < x < 4$$

$$\Rightarrow 2 < |x| < 4$$

$$6 < 3x < 12$$

$$4 < x^2 < 16$$

$$|x^2 + 3x + 9| < |x^2| + |3x| + |9| < |16| + |12| + 9$$

$$< 37$$

$$\delta = \min(1, \frac{\varepsilon}{37})$$

إذا أعطينا  $\varepsilon > 0$  حاول إيجاد  $\delta > 0$  متبناه  $\delta = \frac{\varepsilon}{37}$  فننا

$$|x - 3| < \frac{\varepsilon}{37} \Rightarrow 37|x - 3| < \varepsilon \Rightarrow |x^2 + 3x + 9| |x - 3| < \varepsilon$$

$$\Rightarrow |27 - x^3| < 3$$

✗



(4)  $\lim (\frac{1}{x^2} - 2) = -\frac{7}{4}$

$f(x) = \frac{1}{x^2} - 2$        $D_f = \mathbb{R} - \{0\} = \mathbb{R}^*$

$\forall \varepsilon > 0 \exists \delta > 0 \ni \forall x \in \mathbb{R}^*$       المظهر هو ابتداءه

$|x - 2| < \delta \Rightarrow |\frac{1}{x^2} - 2 + \frac{7}{4}| < \varepsilon$

$|x - 2| < \delta \Rightarrow |\frac{1}{x^2} - \frac{1}{4}| < \varepsilon$       كما نريد أن نجد  $\delta$  من  $\varepsilon$  فقط

$|\frac{1}{x^2} - \frac{1}{4}| = |\frac{4 - x^2}{4x^2}| = \frac{1}{4} \cdot \frac{1}{x^2} |x^2 - 4|$

$= \frac{1}{4} \cdot \frac{1}{x^2} |x+2| |x-2| \rightarrow \textcircled{1}$

$[2 - \delta \quad 2 + \delta]$       نأخذ الجوار  $\delta \leq 1$

$1 < x < 3 \Rightarrow |x+2| \leq |x| + |2| < 5$

$1 < x^2 < 9 \Rightarrow \frac{1}{x^2} < 1$

$\therefore \textcircled{1} \Rightarrow |\frac{1}{x^2} - \frac{1}{4}| < \frac{1}{4} \cdot 1 \cdot \delta = \frac{\delta}{4}$

$\therefore \delta = \min(1, \frac{4\varepsilon}{5})$

إذاً أعطينا  $\varepsilon > 0$  كما دللنا  $\delta > 0$  متبني

$|x - 2| < \frac{4\varepsilon}{5} \Rightarrow \frac{5}{4} |x - 2| < \varepsilon \Rightarrow \frac{1}{4} \cdot \frac{1}{x^2} |x+2| |x-2| < \varepsilon$

$|\frac{4 - x^2}{4x^2}| < \varepsilon \Rightarrow |\frac{1}{x^2} - \frac{1}{4}| < \varepsilon$  #



$$\textcircled{5} \lim_{x \rightarrow 4} \frac{x}{x+3} = \frac{4}{7}$$

$$D_f = \mathbb{R} - \{3\}$$

$$x-3 > 0 \quad 35 > 0 \Rightarrow \forall x \in \mathbb{R} - \{3\} : 0 < |x-4| < \delta \Rightarrow \left| \frac{x}{x+3} - \frac{4}{7} \right| < \epsilon$$

$$|x-4| < \left| \frac{x}{x+3} - \frac{4}{7} \right|$$

نکات اول اینجاست که  $\delta$  به قدری را

$$\left| \frac{x}{x+3} - \frac{4}{7} \right| = \frac{7x - 4x - 12}{7x + 21} = \frac{3x - 12}{7x + 21}$$

$$\Rightarrow \frac{1}{7} \times 3 \times \left| \frac{1}{x+3} \right| \times |x-4| \rightarrow \textcircled{1}$$

ماضی الحوا

$$]4-\delta, 4+\delta[ , \delta < 7$$

$$3 < x < 5 \Rightarrow 6 < |x+3| < 8 \Rightarrow \frac{1}{6} > \frac{1}{|x+3|} > \frac{1}{8}$$

$$\left| \frac{x}{x+3} - \frac{4}{7} \right| = \frac{1}{7} \times 3 \times \frac{1}{6} = \frac{3}{42} = \frac{1}{14}$$

$$\therefore \delta = 14\epsilon \Rightarrow \delta = \min(1, 14\epsilon)$$

$$|x-4| < \delta < 14\epsilon \Rightarrow \frac{|x-4|}{14} < \epsilon$$

$$\frac{1}{7} \cdot 3 \cdot \frac{1}{6} |x-4| < \epsilon \Rightarrow \frac{1}{7} \cdot 3 \cdot \frac{1}{|x+3|} |x-4| < \epsilon$$

$$\left| \frac{7x - 4x - 12}{7x + 21} \right| < \epsilon \Rightarrow \left| \frac{x}{x+3} - \frac{4}{7} \right| < \epsilon$$

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⑦  $\lim_{x \rightarrow 1} \frac{1}{2x-1} = 1$

$$f(x) = \frac{1}{2x-1} \quad D_f = \mathbb{R} - \left\{ \frac{1}{2} \right\}$$

$$\forall \varepsilon > 0 \exists \delta_{\varepsilon} > 0 \Rightarrow \forall x \in \mathbb{R} - \{ \frac{1}{2} \}: \quad \sim |x - \frac{1}{2}| < \delta_{\varepsilon} \Rightarrow |f(x) - 1| < \varepsilon$$

$$0 < |x - 1| < \delta \Rightarrow \left| \frac{1}{x-1} - 1 \right| < \varepsilon$$

۴- ایجاد ال ۸ ج - بخار ایجاد عرق می شود

$$|x-1|, \left| \frac{1}{2x-1} \right|$$

$$\left| \frac{1}{2x_1} - 1 \right| = \left| \frac{1-2x_1}{2x_1-1} \right| = \left| \frac{2-2x_1}{2x_1-1} \right| = 2 \frac{(x-1)}{(2x-1)}$$

نافذة الجوار  $\delta \leq \frac{7}{4}$   $[1-\delta, 1+\delta]$

$$\frac{3}{4} < x < \frac{5}{4} \Rightarrow \frac{3}{2} < 2x < \frac{5}{2} \Rightarrow \frac{1}{2} < 2x-1 < \frac{3}{2}$$

$$2 > \frac{1}{1-2x-11} > \frac{2}{3} \quad 2 \cdot 2 \cdot 8 = 48 \quad \sqrt{5 - \frac{9}{4}}$$

$$|x-1| < \frac{\varepsilon}{4} \Rightarrow 4|x-1| < \varepsilon \Rightarrow 2 \cdot 2|x-1| < \varepsilon$$

$$\left| \frac{1}{2x-1} - 1 \right| < \epsilon \Leftrightarrow \left| \frac{2x-2}{2x-1} \right| < \epsilon \Leftrightarrow 2 \cdot \frac{1}{|2x-1|} |x-1| < \epsilon$$



(٢٢)

$$\textcircled{8} \lim_{x \rightarrow 9} \sqrt{x} = 3$$

$$f(x) = \sqrt{x} \quad D_f = [0, \infty[$$

المطلوب إثباته أن:  $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \Rightarrow \forall x \in [0, \infty[$ ؛

$$0 < |x-9| < \delta \Rightarrow |\sqrt{x}-3| < \varepsilon$$

٢- إيجاد الـ  $\delta$  = إيجاد علاقة بين المقدارين

$$|x-9| < \delta \quad | \quad |\sqrt{x}-3|$$

$$|\sqrt{x}-3| = \frac{|\sqrt{x}+3|}{|\sqrt{x}+3|} \cdot |\sqrt{x}-3| = \frac{|x-9|}{|\sqrt{x}+3|} = \frac{1}{|\sqrt{x}+3|} \cdot |x-9| \quad \textcircled{1}$$

$$]9-\delta, 9+\delta[$$

$$\delta \leq 1$$

ناضراً الجوار

$$8 < x < 10 \Rightarrow \sqrt{8} < \sqrt{x} < \sqrt{10} \Rightarrow \sqrt{8}+3 < \sqrt{x}+3 < \sqrt{10}+3$$

$$\frac{1}{\sqrt{8}+3} > \frac{1}{\sqrt{x}-3} > \frac{1}{\sqrt{10}+3}$$

$$\delta = \min(1, (3+\sqrt{8})\varepsilon) \quad \left| \begin{array}{l} \delta = \varepsilon \\ \delta = (3+\sqrt{8})\varepsilon \end{array} \right.$$



$$\lim_{x \rightarrow 4} \frac{1}{x^2} = \frac{1}{16}$$

Ans

$$f(x) = \frac{1}{x^2} \Rightarrow D_f = \mathbb{R} - \{0\}$$

① ایجاد مجال

$$\forall \varepsilon > 0 \exists \delta_{(\varepsilon)} > 0 \Rightarrow \forall x \in \mathbb{R} - \{0\} : 0 < |x-4| < \delta \Rightarrow \left| \frac{1}{x^2} - \frac{1}{16} \right| < \varepsilon$$

$$|x-4| < \delta \Rightarrow \left| \frac{1}{x^2} - \frac{1}{16} \right| < \varepsilon$$

$$\left| \frac{1}{x^2} - \frac{1}{16} \right| = \frac{|16 - x^2|}{|16x^2|} = \frac{|x-4| |x+4|}{|16x^2|} = |x-4| \cdot \frac{|x+4|}{|16x^2|} \rightarrow *$$

$$]4-\delta, 4+\delta[$$

$$\delta \leq 1$$

\* ناخته اجواب

$$3 < x < 5 \Rightarrow 16 \times 9 < 16x^2 < 16 \times 25 \Rightarrow 144 < 16x^2 < 400$$

$$\Downarrow$$

$$7 < |x+4| < 9$$

$$\frac{9}{400} \delta = \varepsilon \Rightarrow \delta = \frac{400\varepsilon}{9} \quad \varepsilon = 44,4\varepsilon$$

$$|x-4| < \delta < \frac{400\varepsilon}{9} \Rightarrow 9|x-4| < 400\varepsilon \Rightarrow \frac{9}{400}|x-4| < \varepsilon$$

$$\frac{|x+4|}{|16x^2|} |x-4| < \varepsilon \Rightarrow \frac{|16-x^2|}{|16x^2|} \Leftrightarrow \left| \frac{1}{x^2} - \frac{1}{16} \right| < \varepsilon$$



$$10) \lim_{x \rightarrow \frac{1}{2}} (1-2x) = 0$$

$$f(x) = 1-2x, D_f = \mathbb{R}$$

① ایجاد مجال

$$\forall \varepsilon > 0 \exists \delta_0 > 0 \Rightarrow \forall x \in \mathbb{R} : 0 < |x - \frac{1}{2}| < \delta \Rightarrow |1-2x| < \varepsilon$$

$$|1-2x| < \varepsilon \quad |x - \frac{1}{2}| < \frac{\varepsilon}{2}$$

$$|1-2x| = |2x-1| = 2|x - \frac{1}{2}| \quad \begin{matrix} 2\delta = \varepsilon \\ \delta = \frac{\varepsilon}{2} \end{matrix}$$

تعار  $\delta = \frac{\varepsilon}{2}$

$$\sim |x - \frac{1}{2}| < \delta < \frac{\varepsilon}{2} \Rightarrow \sim |x - \frac{1}{2}| < \frac{\varepsilon}{2} \Rightarrow 2|x - \frac{1}{2}| < \varepsilon$$

$$\sim |2x-1| < \varepsilon \Rightarrow \boxed{|1-2x| < \varepsilon} \quad \#$$

$$11) \lim_{x \rightarrow 2} (x^2+x+1) = 7$$

$$f(x) = x^2+x+1, D_f = \mathbb{R}$$

① ایجاد مجال

$$\forall \varepsilon > 0 \exists \delta_0 > 0 \Rightarrow x \in \mathbb{R} : 0 < |x-2| < \delta \Rightarrow |x^2+x+1-7| < \varepsilon$$

$$\Rightarrow |x^2+x-6| < \varepsilon$$

$$|x^2+x-6| < \varepsilon$$

② ایجاد  $\delta$  خاد ایجاد مقیاس

$$|x^2+x-6| = |(x-2)(x+3)| \quad \begin{matrix} |x-2| < \delta \\ |x+3| \end{matrix}$$

$$\exists 2-\delta, 2+\delta$$

ناض الجوار  $\delta \leq 1$

$$1 < x < 3$$

$$4 < |x+3| < 6 \quad \text{③} \quad 6\delta = \varepsilon \quad \sim \delta = \frac{\varepsilon}{6}$$



$$\delta = \frac{\epsilon}{6} \Rightarrow 1 < x-2 < \frac{\epsilon}{6} \Rightarrow 6 < x-2 < \epsilon$$

$$|(x+3)(x-2)| < \epsilon \Rightarrow |(x^2+x-6)| < \epsilon$$

$$(12) \lim_{x \rightarrow -2} (x^2 - \frac{1}{2}) = \frac{7}{2}$$

$$f(x) = x^2 - \frac{1}{2} \quad D_f = \mathbb{R}$$

۱- ایجاد مجال

۲- مطلوبه اثباته

$$\forall \epsilon > 0 \exists \delta_{(\epsilon)} > 0 \ni x \in \mathbb{R} : 0 < |x+2| < \delta \Rightarrow |x^2 - \frac{1}{2} - \frac{7}{2}| < \epsilon \Rightarrow |x^2 - 4| < \epsilon$$

۳- ایجاد ال  $\delta$  فاول ایجاد علاقة بين  $|x^2 - 4|$  و  $|x+2|$

$$|x^2 - 4| = |(x+2)(x-2)| \rightarrow (*)$$

ناخذ الجواب  $\delta \leq 1$

$$]2-\delta, 2+\delta[$$

$$1 < x < 3 \Rightarrow -1 < x-2 < 1$$

$\delta = \epsilon$

$$(13) \lim_{x \rightarrow 2} \frac{1}{x^3} = \frac{1}{8}$$

$$f(x) = \frac{1}{x^3} \quad D_f = \mathbb{R} - \{0\}$$

۱- ایجاد مجال

۲- مطلوبه اثباته

$$\forall \epsilon > 0 \exists \delta_{(\epsilon)} > 0 \ni x \in \mathbb{R} - \{0\} : 0 < |x-2| < \delta \Rightarrow \left| \frac{1}{x^3} - \frac{1}{8} \right| < \epsilon$$

۳- ایجاد ال  $\delta$  فاول ایجاد علاقة بين  $\left| \frac{1}{x^3} - \frac{1}{8} \right|$  و  $|x-2|$



$$\left| \frac{1}{x^3} - \frac{1}{8} \right| = \left| \frac{8 - x^3}{8x^3} \right| = \left| \frac{x^3 - 8}{8x^3} \right| = \left| \frac{(x-2)(x^2+2x+4)}{8x^3} \right|$$

$$= |x-2| \cdot \frac{|x^2+2x+4|}{8x^3} \quad (*)$$

$$]2-8, 2+8[ \quad 1 < x^3 < 27 \Rightarrow 8 < 8x^3 < 216$$

$$1 < x < 3 \quad 1 < x^2 < 9$$

$$2 < 2x < 6$$

$$3 < \frac{1}{2}x+1 < 7$$

$$\delta \cdot \frac{|x+7|}{\frac{1}{8}} = (16)(8)\delta = \varepsilon$$

$$\delta = \frac{\varepsilon}{128}$$

نادر الجول 857

$$\left( \frac{1}{8} \right) \Rightarrow \frac{1}{8+x} > \frac{1}{216}$$

$$14) \lim_{x \rightarrow -3} \frac{1}{1+x^3} = -\frac{1}{26}$$

$$f(x) = \frac{1}{1+x^3} \Rightarrow D_f = \mathbb{R} - \{-1\}$$

1- إيجاد مجال

$$x \in \mathbb{R} \Rightarrow x \neq -1$$

$$x \in \mathbb{R} \Rightarrow x \neq -1$$

$$| \frac{1}{1+x^3} + \frac{1}{26} | < \varepsilon$$

$$|x+3| \neq | \frac{1}{1+x^3} + \frac{1}{26} |$$

$$\left| \frac{26+1+x^3}{26+26x^3} \right| = \frac{|x^3+27|}{26|x^3+1|} = \frac{|x+3||x^2-3x+9|}{26|x^3+1|}$$

2- إيجاد  $\delta$  كدالة لـ  $\varepsilon$  (مقابل  $\varepsilon$ )



10.  $\lim_{x \rightarrow 4} \sqrt{x} = 2$

۱۵ ایجاد اطفال

$$f(x) = \sqrt{x} \Rightarrow D_f = [0, \infty[$$

$f(x) = \sqrt{x} \Rightarrow D_f = [0, \infty)$   
 $\forall \varepsilon > 0 \exists \delta_{(x)} > 0 \exists x \in [0, \infty) : 0 < |x - 4| < \delta_{(x)} \Rightarrow f(x) \neq 2$

$$\Rightarrow |\sqrt{x} - 2| < \varepsilon$$

(2) إيجاد الـ  $\delta$  : حاول إيجاد عدد  $\delta$  مقداره  $\delta$   
 $\Rightarrow |\sqrt{x} - 2| < \epsilon$   
 $|\sqrt{x} - 2|$      $|x - 4|$

$$|\sqrt{x}-2| = |\sqrt{x}-2| \cdot \frac{|\sqrt{x}+2|}{|\sqrt{x}+2|} = \frac{|x-4|}{|\sqrt{x}+2|} = |x-4| \cdot \frac{1}{|\sqrt{x}+2|} \quad (*)$$

$$]4-8, 4+8[$$

نامہ از اجوار 851

$$3 < x < 5 \Rightarrow \sqrt{3} < \sqrt{x} < \sqrt{5} \Rightarrow 2\sqrt{2}\sqrt{x} + 2 < 2 + \sqrt{5}$$

$$\frac{1}{2+\sqrt{3}} > \frac{1}{|\sqrt{x}+2|} > \frac{1}{2+\sqrt{5}} \Rightarrow \frac{\delta}{2+\sqrt{3}} = \varepsilon \quad \sim \delta = (2+\sqrt{3})\varepsilon$$

$$\textcircled{16} \lim_{x \rightarrow 1} \frac{x}{x+2} = \frac{1}{3}$$

$$f(x) = \frac{x}{x+2} \quad \Rightarrow D_f = \mathbb{R} - \{-2\}$$

10/5/2021

$$\forall \varepsilon > 0 \exists \delta_{(\varepsilon)} > 0 \exists x \in \mathbb{R} - \{-2\} : 0 < |x - 1| < \delta_{(\varepsilon)} \Rightarrow |2x + 4| < \varepsilon$$

$$\Rightarrow \left| \frac{x}{x+2} - \frac{1}{3} \right| < \epsilon$$

$$|\frac{x}{x+2} - \frac{1}{3}| \Rightarrow |x-1|$$

$$\left| \frac{x}{x+2} - \frac{1}{3} \right| = \frac{|3x - x - 2|}{3|x+2|} = \frac{|2x-2|}{3|x+2|} = \frac{2|x-1|}{3|x+2|}$$

$$]1-\delta, 1+\delta[$$

Scribble



$$(17) \lim_{x \rightarrow \infty} \frac{x}{x+2} = 1$$

$$f(x) = \frac{x}{x+2} \Rightarrow D_f = \mathbb{R} - \{-2\}$$

ⓐ إيجاد مجال

ⓑ المطلوب ابتداء من

$$\forall \varepsilon > 0 \exists K_{(\varepsilon)} > 0 \Rightarrow \forall x \in \mathbb{R} - \{-2\}$$

$$: x > K \Rightarrow \left| \frac{x}{x+2} - 1 \right| < \varepsilon$$

$$\left| \frac{x}{x+2} - 1 \right| = \left| \frac{x - x - 2}{x+2} \right| = \left| \frac{-2}{x+2} \right| < \varepsilon \quad \text{ⓐ إيجاد الـ } K$$

$$\Rightarrow \frac{|x+2|}{2} > \frac{1}{\varepsilon}$$

$$x+2 > \frac{2}{\varepsilon}$$

$$x > \frac{2}{\varepsilon} - 2$$

$$\Rightarrow |x+2| \geq \frac{2}{\varepsilon}$$

or  $x+2 < -\frac{2}{\varepsilon}$

$$\left\{ \begin{array}{l} x < -\frac{2}{\varepsilon} - 2 \end{array} \right.$$

$$K = \frac{2}{\varepsilon} - 2$$

$$(18) \lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$$

$$f(x) = \frac{1}{(x-1)^2} \Rightarrow D_f = \mathbb{R} - \{1\}$$

ⓐ إيجاد مجال

$$\forall M > 0 \exists \delta > 0 \Rightarrow \forall x \in \mathbb{R} - \{1\} : 0 < |x-1| < \delta$$

$$\Rightarrow \frac{1}{(x-1)^2} > M$$

$$\frac{1}{(x-1)^2} > M \Rightarrow (x-1)^2 < \frac{1}{M} \Rightarrow |x-1| < \frac{1}{\sqrt{M}} \delta \quad \text{ⓐ إيجاد الـ } \delta$$

$$\delta = \frac{1}{\sqrt{M}}$$



$$19) \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$f(x) = \frac{1}{x} \rightarrow D_f = \mathbb{R} - \{0\} = \mathbb{R}^*$$

$$\forall \varepsilon > 0 \exists K_\varepsilon > 0 \Rightarrow \forall x \in \mathbb{R}^* : x > K$$

$$|\frac{1}{x}| < \varepsilon$$

$$|\frac{1}{x}| < \varepsilon \Rightarrow |x| > \frac{1}{\varepsilon} \Rightarrow x > \frac{1}{\varepsilon} \text{ or } x < -\frac{1}{\varepsilon}$$

$$\text{choose } K = \frac{1}{\varepsilon}$$

$$20) \lim_{x \rightarrow \infty} (x^2 - 1) = \infty$$

$$f(x) = (x^2 - 1) \rightarrow D_f = \mathbb{R}$$

$$\forall M > 0 \exists K_{(M)} > 0 \Rightarrow x \in \mathbb{R} : x < -K \Rightarrow x^2 - 1 > M$$

$$x^2 - 1 > M \Rightarrow x^2 > M + 1 \Rightarrow |x| > \sqrt{M+1} \Rightarrow x < -\sqrt{M+1} \text{ or } x > \sqrt{M+1}$$

$$\boxed{x > \sqrt{M+1}} \text{ or } \boxed{x < -\sqrt{M+1}}$$

$$\therefore \text{choose } K = \sqrt{M+1}$$

$$21) \lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 3} = 1$$

$$f(x) = \frac{x^2}{x^2 - 3} \rightarrow D_f = \mathbb{R} - \{\pm\sqrt{3}\}$$

$$\forall \varepsilon > 0 \exists K > 0 \Rightarrow \forall x \in \mathbb{R} - \{\pm\sqrt{3}\} : x > K$$

$$|\frac{x^2}{x^2 - 3} - 1| < \varepsilon$$





$$21) \left| \frac{x^2}{x^2-3} - 1 \right| = \left| \frac{x^2 - x^2 + 3}{x^2-3} \right| = \left| \frac{3}{x^2-3} \right| < \epsilon$$

$$\Rightarrow \frac{|x^2-3|}{3} > \frac{1}{\epsilon} \Rightarrow |x^2-3| > \frac{3}{\epsilon}$$

$$x^2-3 > \frac{3}{\epsilon}$$

$$x^2 > \frac{3}{\epsilon} + 3$$

$$|x| > \sqrt{\frac{3}{\epsilon} + 3}$$

$$\Rightarrow K = \sqrt{\frac{3}{\epsilon} + 3}$$

$$(ii) x^2-3 < -\frac{3}{\epsilon}$$

$$x^2 < 3 - \frac{3}{\epsilon}$$

$$x > \sqrt{\frac{3}{\epsilon} + 3} \text{ or } x < -\sqrt{\frac{3}{\epsilon} + 3}$$

$$22) \lim_{x \rightarrow \infty} (x+3) = \infty$$

$$f(x) = (x+3) \text{ , } D_f = \mathbb{R}$$

$$\forall M > 0 \exists K_M > 0 \Rightarrow \forall x \in \mathbb{R} : x > K \Rightarrow x+3 > M$$

$$x+3 > M \Rightarrow x > M+3$$

$$\text{choose } K = M+3$$

$$23) \lim_{x \rightarrow 0} \frac{1}{|x|} = \infty$$

$$f(x) = \frac{1}{|x|}$$

$$D_f = \mathbb{R} - \{0\} = \mathbb{R}^*$$

$$\forall M > 0 \exists \delta_M > 0 \Rightarrow \forall x \in \mathbb{R}^* \text{ such that } 0 < |x| < \delta \Rightarrow \frac{1}{|x|} > M$$

$$\Rightarrow |x| < \frac{1}{M} \Rightarrow x < \frac{1}{M} \text{ or } x > -\frac{1}{M}$$

$$\Rightarrow K = \frac{1}{M}$$



$$(24) \lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0$$

$$f(x) = \frac{1}{x^2} \Rightarrow D_f = \mathbb{R} - \{0\} = \mathbb{R}^*$$

ایجاد ۱

$$\forall \varepsilon > 0 \exists K_0 > 0 \Rightarrow \forall x \in \mathbb{R}^* : x < -K \Rightarrow \frac{1}{x^2} < \varepsilon$$

$$\frac{1}{x^2} < \varepsilon \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow |x| > \frac{1}{\sqrt{\varepsilon}}$$

ایجاد ۲

$$x > \frac{1}{\sqrt{\varepsilon}} \quad \text{or} \quad x < -\frac{1}{\sqrt{\varepsilon}}$$

$$\therefore \text{choose } K = \frac{1}{\sqrt{\varepsilon}}$$

$$(25) \lim_{x \rightarrow 1} \frac{1}{|x-1|} = \infty$$

$$f(x) = \frac{1}{|x-1|} \Rightarrow D_f = \mathbb{R} - \{1\}$$

ایجاد ۱

$$\forall M > 0 \in \mathbb{R}_{(M)} \Rightarrow \forall x \in \mathbb{R} - \{1\}$$

ایجاد ۲

$$0 < |x-1| < \delta \Rightarrow \frac{1}{|x-1|} > M$$

$$\frac{1}{|x-1|} > M \Rightarrow |x-1| < \frac{1}{M}$$

ایجاد ۳

$$x-1 < \frac{1}{M} \quad \text{or} \quad x-1 > -\frac{1}{M}$$

$$x < \frac{1}{M} + 1 \quad \text{or} \quad x > 1 - \frac{1}{M}$$

$$\text{choose } \delta = \frac{1}{M} + 1$$



26  $\lim_{x \rightarrow \infty} \frac{x}{2x+1} = \frac{1}{2}$

$f(x) = \frac{x}{2x+1}$  ,  $D_f = \mathbb{R} - \{-\frac{1}{2}\}$

① إيجاد مجال  $x$   
 ② إيجاد نقطة التوقف

$\forall \varepsilon > 0 \exists K_0 > 0 \Rightarrow x \in \mathbb{R} - \{-\frac{1}{2}\}$

$x > K \Rightarrow \left| \frac{x}{2x+1} - \frac{1}{2} \right| < \varepsilon$

② إيجاد الـ  $K$

$\left| \frac{x}{2x+1} - \frac{1}{2} \right| = \left| \frac{2x - 2x - 1}{2(2x+1)} \right| = \left| \frac{-1}{2(2x+1)} \right| = \left| \frac{1}{4x+2} \right| < \varepsilon$

$\Rightarrow |4x+2| > \frac{1}{\varepsilon}$

$4x+2 > \frac{1}{\varepsilon}$

①

$4x+2 < -\frac{1}{\varepsilon}$

②

$4x < -2 - \frac{1}{\varepsilon}$

$x > \frac{\frac{1}{\varepsilon} - 2}{4}$

③

$x < \frac{-2 - \frac{1}{\varepsilon}}{4}$

$\Rightarrow$  choose  $K = \frac{1}{4} \left( \frac{1}{\varepsilon} - 2 \right) = \frac{1}{4\varepsilon} - \frac{1}{2}$

★ Notes

①  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

②

$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$

③

$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

④  $1 - \cos x = 2 \sin^2 \left( \frac{x}{2} \right)$     ⑤  $\sin(n\pi) = 0 \quad n \in \mathbb{Z}$

$\cos(n\pi) = (-1)^n$     ⑥  $\cos x - \cos y = -2 \sin \left( \frac{1}{2}(x+y) \right) \sin \left( \frac{1}{2}(x-y) \right)$

$\sin x - \sin y = 2 \sin \left( \frac{1}{2}(x-y) \right) \cos \left( \frac{1}{2}(x+y) \right)$

$\cos 2\theta = 1 - \sin^2 \theta$

$\cos \theta = 1 - \sin^2 \frac{\theta}{2}$



$$D. \lim \sin 4x$$

$$(24) \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$

$$f(x) = \frac{1}{x^2} \Rightarrow D_f = \mathbb{R} - \{0\} = \mathbb{R}^*$$

$$\forall \varepsilon > 0 \exists K_0 > 0 \Rightarrow \forall x \in \mathbb{R}^* : x < -K \Rightarrow \frac{1}{x^2} < \varepsilon$$

$$\frac{1}{x^2} < \varepsilon \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow |x| > \frac{1}{\sqrt{\varepsilon}}$$

$$\boxed{x > \frac{1}{\sqrt{\varepsilon}}} \text{ or } x < -\frac{1}{\sqrt{\varepsilon}}$$

$$\text{choose } K = \frac{1}{\sqrt{\varepsilon}}$$

$$(25) \lim_{x \rightarrow 7} \frac{1}{|x-1|} = \infty$$

$$f(x) = \frac{1}{|x-1|} \Rightarrow D_f = \mathbb{R} - \{1\}$$

$$\forall n > 0 \in \mathbb{N} \exists \delta > 0 \Rightarrow \forall x \in \mathbb{R} - \{1\}$$

$$0 < |x-1| < \delta \Rightarrow \frac{1}{|x-1|} > n$$

$$\frac{1}{|x-1|} > n \Rightarrow |x-1| < \frac{1}{n}$$

$$x-1 < \frac{1}{n} \text{ or } x-1 > -\frac{1}{n}$$

$$x < \frac{1}{n} + 1 \text{ or } x > 1 - \frac{1}{n}$$

$$\text{choose } \delta = \frac{1}{n} + 1$$



$$2) \lim_{x \rightarrow 0} \sin 4x$$

$$1) \lim_{x \rightarrow 0} \frac{\sin 4x}{5x}$$

Ans

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{5x}$$

$\Rightarrow$

$$\frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 4x}{x}$$

$$\frac{4}{5} \lim_{x \rightarrow 0} \frac{\sin 4x}{4x}$$

$$\frac{4}{5} \times 1 = \frac{4}{5}$$

$$2) \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2-1}$$

Ans

$$\lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)(x+1)}$$

$\Rightarrow$

$$\frac{1}{x+1} \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)}$$

$$\Rightarrow \frac{1}{1+1} \times 1 = \frac{1}{2}$$

$$3) \lim_{x \rightarrow 4} \frac{\tan(2-\sqrt{x})}{x-4}$$

Ans

$$\lim_{x \rightarrow 4} \frac{\tan(2-\sqrt{x})}{(\sqrt{x}-2)(\sqrt{x}+2)}$$

$\Rightarrow$

$$\lim_{x \rightarrow 4} \frac{\tan(2-\sqrt{x})}{-(2-\sqrt{x})(\sqrt{x}+2)}$$

$\Rightarrow$

$$\frac{1}{-(\sqrt{x}+2)} \lim_{2-\sqrt{x} \rightarrow 0} \frac{\tan 2-\sqrt{x}}{2-\sqrt{x}}$$

$$\Rightarrow \frac{1}{-(\sqrt{4}+2)} \times 1 = -\frac{1}{4}$$

$$4) \lim_{x \rightarrow -5} \frac{\sin(x^2-25)}{x+5}$$

Ans

$$\lim_{x \rightarrow -5} \frac{\sin x^2-25}{x+5} \times \frac{x-5}{x-5}$$

$\Rightarrow$

$$\lim_{x \rightarrow -5} \frac{\sin(x^2-25)}{x^2-25} \cdot (x-5)$$

$$x \rightarrow -5$$

$$x^2 \rightarrow 25$$

$$x^2-25 \rightarrow 0$$

$$\lim_{x^2-25 \rightarrow 0} \frac{\sin(x^2-25)}{(x^2-25)} \cdot \lim_{x \rightarrow -5} x-5$$

$\Rightarrow$

$$1 \cdot (-5-5) = -10$$



$$5 \quad \lim_{x \rightarrow 0} \frac{2 - 2 \cos(5x)}{3x^2}$$

Ans

$$\lim_{x \rightarrow 0} \frac{2(1 - \cos(5x))}{3x^2} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{2 \sin^2(\frac{5}{2}x)}{x^2}$$

$$\left\{ \frac{4}{3} \lim_{x \rightarrow 0} \frac{\sin \frac{5}{2}x}{x} \cdot \frac{\sin \frac{5}{2}x}{x} = \frac{4}{3} \lim_{x \rightarrow 0} \frac{\frac{5}{2}}{\frac{5}{2}x} \frac{\sin \frac{5}{2}x}{\frac{5}{2}x} \cdot \lim_{x \rightarrow 0} \frac{\frac{5}{2} \sin \frac{5}{2}x}{\frac{5}{2}x} \right.$$

$$= \frac{4}{3} \times \frac{5}{2} \times \frac{5}{2} = \frac{25}{3}$$

$$6 \quad \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$$

Ans

$$\lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\sin x \left( \frac{1}{\cos x} - 1 \right)}{x^3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{\cos x}}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 \cos x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2 \cos x} \Rightarrow 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin \frac{1}{2}x}{\frac{1}{2}x}$$

$$\times \lim_{x \rightarrow 0} \frac{\sin \frac{1}{2}x}{\frac{1}{2}x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x}$$

$$2 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}$$



$$7) \lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 7x}$$

(Ans)

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 7x} = \lim_{x \rightarrow 0} \frac{\frac{\sin 4x}{x}}{\frac{\sin 7x}{x}} = \lim_{x \rightarrow 0} \frac{4 \frac{\sin 4x}{4x}}{7 \frac{\sin 7x}{7x}}$$

$$\frac{4}{7} \cdot \frac{\lim_{4x \rightarrow 0} \frac{\sin 4x}{4x}}{\lim_{7x \rightarrow 0} \frac{\sin 7x}{7x}} = \frac{4}{7} \cdot \frac{1}{1} = \frac{4}{7}$$

$$8) \lim_{x \rightarrow \pi} \frac{\sin(4x)}{\sin(7x)}$$

(Ans)

$$\text{let } x - \pi = y \quad \boxed{y + \pi = x}$$

$$\lim_{y \rightarrow 0} \frac{\sin(4y + 4\pi)}{\sin(7y + 7\pi)} = \lim_{y \rightarrow 0} \frac{\sin(4y) \cos(4\pi) + \cos(4y) \sin(4\pi)}{\sin(7y) \cos(7\pi) + \cos(7y) \sin(7\pi)}$$

$$\lim_{y \rightarrow 0} \frac{\sin 4y}{\sin 7y} = \lim_{y \rightarrow 0} \frac{\frac{\sin 4y}{y}}{\frac{\sin 7y}{y}} = \frac{4}{7} \cdot \frac{\lim_{4y \rightarrow 0} \frac{\sin 4y}{4y}}{\lim_{7y \rightarrow 0} \frac{\sin 7y}{7y}}$$

$$= \frac{4}{7} \cdot \frac{1}{1} = \frac{4}{7}$$

$$\lim_{x \rightarrow 0} \frac{x \sin\left(\frac{4}{x^2}\right)}{1 + x^2}$$

(Ans)

$$\lim_{x \rightarrow 0} \frac{x}{1+x^2} \cdot \sin\left(\frac{4}{x^2}\right) = 0$$

والمتوسط والحدود



$$10) \lim_{x \rightarrow \infty} \frac{\sin x}{x}$$

Ans

$$\lim_{x \rightarrow \infty} \frac{1}{x} \cdot \sin x \quad \text{is } \lim = 0$$

$$11) \lim_{x \rightarrow \infty} \frac{x \sin x - \cos^2 x}{x^2 + x}$$

Ans

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$$\lim_{y \rightarrow 0} \frac{\frac{1}{y} \sin(\frac{1}{y}) - \cos^2(\frac{1}{y})}{(\frac{1}{y})^2 + \frac{1}{y}}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\frac{1}{y^2} \sin(\frac{1}{y}) - \cos^2(\frac{1}{y})}{\frac{1+y}{y^2}} = \lim_{y \rightarrow 0} \frac{y \sin(\frac{1}{y}) - y^2 \cos^2(\frac{1}{y})}{1+y}$$

$$\lim_{y \rightarrow 0} \frac{y}{1+y} \cdot \sin(\frac{1}{y}) - \lim_{y \rightarrow 0} \frac{y^2}{1+y} \cdot \cos^2(\frac{1}{y})$$

$$\text{is } \lim = 0$$

$$12) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}$$

Ans

$$\lim_{x \rightarrow a} \frac{2 \sin(\frac{1}{2}(x-a)) \cdot \cos(\frac{1}{2}(x+a))}{x-a}$$

$$2 \lim_{x \rightarrow a} \frac{\sin(\frac{1}{2}(x-a))}{\frac{1}{2}(x-a)} \cdot \cos(\frac{1}{2}(x+a))$$

$$2 \cdot \frac{1}{2} \cdot 1 \cdot \cos \frac{1}{2}(2a) = \cos a$$



$$(13) \lim_{x \rightarrow a} \frac{\cos^3 x - \cos^3 a}{x - a}$$

[Ans]

$$\lim_{x \rightarrow a} \frac{(\cos x - \cos a)(\cos^2 x + \cos x \cos a + \cos^2 a)}{x - a}$$

$$\lim_{x \rightarrow a} \left( \frac{\frac{1}{2} - 2 \sin\left(\frac{1}{2}(x-a)\right)}{\frac{1}{2}(x-a)} \right) \cdot \sin\left(\frac{1}{2}(x+a)\right) (\cos^2 x + \cos x \cos a + \cos^2 a)$$

$$1 - 2 \times \frac{1}{2} \times 1 \times \sin\left(\frac{1}{2}(2a)\right) \times (\cos^2 a + \cos^2 a + \cos^2 a) \\ = (-1)(\sin a)(3\cos^2 a) \\ = -3 \sin a \cos^2 a$$

$$(14) \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x} - 2}$$

[Ans]

$$\lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x} - 2} \times \frac{(\sqrt{2x+1} + 3)}{(\sqrt{2x+1} + 3)} \times \frac{(\sqrt{x} + 2)}{(\sqrt{x} + 2)}$$

$$\lim_{x \rightarrow 4} \frac{(2x+1-9)(\sqrt{x}+2)}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x \rightarrow 4} \frac{2(x-4)(\sqrt{x}+2)}{(x-4)(\sqrt{2x+1}+3)}$$

~~$$\frac{(2)(4) + 1 - 9}{(4-4)(\sqrt{4}+3)}$$~~

$$= \frac{2(\sqrt{4}+2)}{\sqrt{2(4)+1}+3} = \frac{2(2+2)}{3+3} = \frac{8}{6}$$



15)  $\lim_{x \rightarrow 8} \frac{\sqrt{2x+9} - 5}{\sqrt{x} - 2}$

[Ans]

$$\lim_{x \rightarrow 8} \frac{(\sqrt{2x+9} - 5)}{(\sqrt{x} - 2)} \cdot \frac{(\sqrt{2x+9} + 5)}{(\sqrt{2x+9} + 5)} = \frac{(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}{(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}$$

$$\lim_{x \rightarrow 8} \frac{(2x-16) \cdot 2(8)}{(2x+9-25)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)} = \frac{2(16-16) \cdot 2(8)}{(2x+9-25)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}$$

$$\lim_{x \rightarrow 8} \frac{2(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}{(\sqrt{2x+9} + 5)}$$

$$\frac{2(16)^{\frac{2}{3}} + 2(8)^{\frac{1}{3}} + 4}{\sqrt{2 \times 8 + 9} + 5} = \frac{12}{5}$$

$$\sqrt{2 \times 8 + 9} + 5$$

$$5 + \frac{12}{5}$$

16)  $\lim_{x \rightarrow 2} \frac{x^3 + x^2 - 12}{x - 2}$

[Ans]

$$\lim_{x \rightarrow 2} \frac{x^3 - 8 + x^2 - 4}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4) + (x-2)}{(x-2)}$$

$$\lim_{x \rightarrow 2} (x^2 + 2x + 4) + (x + 2)$$

$$4 + 4 + 4 + 4 = 16$$



$$(17) \lim_{x \rightarrow \infty} \frac{\sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}}{\sqrt{2x+3}}$$

Ans

$$\lim_{x \rightarrow \infty} \frac{\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}}}{\frac{(2x+3)^{\frac{1}{2}}}{x^{\frac{1}{2}}}} \Rightarrow \lim_{x \rightarrow \infty} \frac{1 + x^{-\frac{1}{6}} + x^{-\frac{3}{4}}}{(2 + \frac{3}{x})^{\frac{1}{2}}}$$

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^{\frac{1}{6}}} + \frac{1}{x^{\frac{3}{4}}}}{(2 + \frac{3}{x})^{\frac{1}{2}}} \Rightarrow \frac{1 + 0 + 0}{(2+0)^{\frac{1}{2}}} = \frac{1}{(2)^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

$$(18) \lim_{x \rightarrow \pi} \frac{\pi - x}{\sin x}$$

Ans let  $y = \pi - x$   $\therefore x = \pi - y$

$$\lim_{y \rightarrow 0} \frac{\pi - (\pi - y)}{\sin(\pi - y)} = \lim_{y \rightarrow 0} \frac{y}{\sin y \cos \pi} = \lim_{y \rightarrow 0} \frac{y}{\sin y} \cdot \frac{1}{(-1)}$$

$$\lim_{y \rightarrow 0} \frac{y}{\sin y} = 1$$

$$(19) \lim_{x \rightarrow -2} \frac{\sin(\pi x)}{x+2}$$

$$\begin{aligned} x &\rightarrow -2 \\ \pi x &\rightarrow -2\pi \\ y &= \pi x + 2\pi \\ x &= \frac{y - 2\pi}{\pi} \end{aligned}$$

Ans

$$\lim_{y \rightarrow 0} \frac{\sin(\pi \frac{y-2\pi}{\pi})}{\frac{y-2\pi}{\pi} + 2} \Rightarrow \lim_{y \rightarrow 0} \frac{\sin(y-2\pi)}{\frac{y-2\pi+2\pi}{\pi}} \Rightarrow \lim_{y \rightarrow 0} \frac{\sin y \cos 2\pi}{y}$$

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} \Rightarrow \pi \cdot 1 = \pi$$



20)  $\lim_{x \rightarrow \frac{\pi}{2}} (\frac{\pi}{2} - x) \tan x$

$$\lim_{x \rightarrow \frac{\pi}{2}} (\frac{\pi}{2} - x) \frac{\sin x}{\cos x}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \sin x \cdot \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\cos x}$$

$$1 \cdot 1 = 1$$

Handwritten notes and diagrams:

- A small diagram showing a right triangle with angle  $x$ , hypotenuse 1, and sides  $\sin x$  and  $\cos x$ .
- A note:  $\cos x \rightarrow 0$  as  $x \rightarrow \frac{\pi}{2}$ .
- A boxed note:  $1 - \cos^2 x = \sin^2 x$ .
- A note:  $\sin x \rightarrow 1$  as  $x \rightarrow \frac{\pi}{2}$ .

21)  $\lim_{x \rightarrow 0} \frac{\sqrt{x} - 1}{x}$

Ans

$$\lim_{x \rightarrow 0} \frac{\sqrt{x} - 1}{x} \times \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \rightarrow 0} \frac{x - 1}{x(\sqrt{x} + 1)}$$

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

22)  $\lim_{x \rightarrow 0} \frac{1 - \cos x^4}{1 - \cos x^2}$

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

$$\frac{1 - \cos x^4}{1 - \cos x^2} = \frac{2 \sin^2 \frac{x^4}{2}}{2 \sin^2 \frac{x^2}{2}}$$

$$= \lim_{x \rightarrow 0} \left( \frac{\sin^2 \frac{x^4}{2}}{\sin^2 \frac{x^2}{2}} \right) = \lim_{x \rightarrow 0} \left( \frac{\sin \frac{x^4}{2}}{\sin \frac{x^2}{2}} \cdot \frac{x^2}{x^2} \right)$$

$$= 0$$



$$23 \lim_{x \rightarrow \infty} x^2 (1 - \cos(\frac{1}{x})) \quad \text{---} \lim = 0$$

$$24 \lim_{x \rightarrow a} \frac{\sin x^2 - \sin a^2}{x - a}$$

Ans

$$\lim_{x \rightarrow a} \frac{2 \sin(\frac{1}{2}(x^2 - a^2)) \cdot \cos(\frac{1}{2}(x^2 + a^2))}{(x - a)} \times \frac{\frac{1}{2}(x+a)}{\frac{1}{2}(x+a)}$$

$$2 \cdot \frac{1}{2}(x+a) \lim_{x \rightarrow a} \frac{\sin(\frac{1}{2}(x^2 - a^2))}{\frac{1}{2}(x^2 - a^2)} \quad \lim_{x \rightarrow a} \sin(\frac{1}{2}(x^2 + a^2))$$

$\frac{1}{2}(x^2 - a^2) \rightarrow 0$

$$\therefore 2a \cos a^2 \quad \checkmark$$

$$25 \lim_{x \rightarrow -2} \frac{\tan(4 - x^2)}{x^2 + 3x - 10}$$

Ans

$$\lim_{x \rightarrow -2} \frac{\tan(4 - x^2)}{(x-2)(x+5)} \Rightarrow \frac{1}{x+5} \lim_{x \rightarrow -2} \frac{\tan(4 - x^2)}{x-2} \times \frac{x+2}{x+2}$$

$$\frac{x+2}{5} \lim_{x \rightarrow -2} \frac{\tan(4 - x^2)}{x^2 - 4} \Rightarrow \frac{-(x+2)}{x+5} \lim_{x \rightarrow -2} \frac{\tan(4 - x^2)}{(4 - x^2)}$$

$$0 \cdot 1 = 0$$

$$x \rightarrow -2 \quad (24)$$



$$26) \lim_{x \rightarrow 0} \frac{\sin^2(mx) + \tan^2(Kx)}{x^2}$$

Ans

$$\lim_{x \rightarrow 0} \frac{\sin^2(mx)}{x^2} + \lim_{x \rightarrow 0} \frac{\tan^2(Kx)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\sin mx}{mx} \cdot \frac{\sin mx}{mx} + \lim_{x \rightarrow 0} \frac{\tan Kx}{Kx} \cdot \frac{\tan Kx}{Kx}$$

$$\lim_{x \rightarrow 0} \frac{\sin^2(mx) + \tan^2(Kx)}{x^2} \times \frac{m^2 K^2}{m^2 K^2}$$

$$m^2 K^2 \lim_{x \rightarrow 0} \frac{\sin^2(mx) + \tan^2(Kx)}{m^2 K^2 x^2}$$

$$m^2 K^2 \left[ \lim_{x \rightarrow 0} \frac{\sin^2(mx)}{m^2 K^2 x^2} + \lim_{x \rightarrow 0} \frac{\tan^2(Kx)}{m^2 K^2 x^2} \right]$$

$$m^2 K^2 \left[ \frac{1}{K^2} \left( \lim_{x \rightarrow 0} \frac{\sin(mx)}{mx} \cdot \lim_{x \rightarrow 0} \frac{\sin(mx)}{mx} \right) + \frac{1}{m^2} \left( \lim_{x \rightarrow 0} \frac{\tan(Kx)}{Kx} \cdot \lim_{x \rightarrow 0} \frac{\tan(Kx)}{Kx} \right) \right]$$

$$m^2 K^2 \left( \frac{1}{K^2} + \frac{1}{m^2} \right) = \boxed{m^2 + K^2}$$

